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1. **Linear differential equation**

   We define the linear form of differential equation as

   $$y' + p(x)y = f(x)$$

   Liner means the the DE is linear in y' and y. Since it is not separable, we imply the method *integrating factor* $r(t)$ which is to construct a total derivative of $r(t) \cdot y$. We want a $r(x)$ satisfying

   $$y' \cdot r(x) + y \cdot r(x)p(x) = f(x)r(t)$$

   where $r(x)p(x) = r'(x)$. Then for convenience, we consider $p(x) = p$ instead, we can get

   $$r(t) = e^{\int p\, dt}$$

   then the DE becomes

   $$\frac{d}{dx}[r(x)y(x)] = f(x) \cdot r(x) \implies r(x)y(x) = \int f(x) \cdot r(x)\ dx$$

   which is solvable.

2. **Non-linear exact differential equation**

   Let form of deferential equation to be

   $$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \implies \quad M(x,y)\ + N(x,y)\,\frac{dy}{dx} = 0$$

   Definition 1. Exact deferential equation *A DE is called exact if there is a potential function $\phi(x,y)$ s.t. $M = \phi_x$ and $N = \phi_y$.*

   Theorem 1. *If $M_y = N_x$, then near any point (x_0, y_0) (locally) there is a function $\phi(x,y)$ so that $\phi_x = M$ and $\phi_y = N$.*

   which generate the way to check whether a DE is exact of not. Notice this does not works globally.

   (a) **Solving the exact DE**

   ① Applying theorem 1 to check the exact-ability of the DE.

   ② Because of the existence of the potential function, let

   $$\phi(x,y) = \int M(x,y)\,dx = Q(x,y) + h(y)$$

   since M is generated from the partial diri. of ϕ, so the integral is w.r.t x and the constant term may include y.

   ③ we get ϕ so far. Then we have

   $$\phi_y(x,y) = \frac{d}{dy}[Q(x,y) + h(y)] = Q_y(x,y) + h'(y) \quad = \quad N(x,y)$$

then h'(y) = N(x,y) - Q_y. Then we know both $Q(x, y)$ and $h(y)$ which gives implicit form of $\phi(x, y)$.

(b) **Case for inexact differential equation**

Similar to the linear DE, we want to find an integration factor $\mu(x, y)$ to construct an exact DE and consequently solve is by process form (a). The DE becomes

$$\mu M(x, y) + \mu N(x, y)\frac{dy}{dx} = 0$$

and in order to make it exact, we need

$$(\mu M)_y = (\mu N)_x \implies \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

which is a PDE, difficult to solve and not aim for this course. So we try $\mu = \mu(x)$ and $\mu = \mu(y)$ which makes several terms above diminishes. Q: Why we care about PDE? What we care about is whether they are equal or not?A: Since we want to use this DE to solve μ.

3. **Autonomous Equation**

Definition 2. Let $x = x(t)$ and $\frac{dx}{dt} = f$. If f is independent from from t, which is $f = f(x)$, then we call $\frac{dx}{dt} = f(x)$ *autonomous equation*. If $f(x_0) = 0$, then x_0 is a *fixed point*, and then $x(t) = x_0$ is a *constant/equilibrium* solution.

Again, remember the solution of the DE is a function x w.r.t t. So here the equilibrium solution is a constant function.
Not finished yet

4. **Second order linear ODE**

The second order ODE is in form of

$$A(x)y'' + B(x)y' + c(x)y = F(x) \longrightarrow cy'' + p(x)y' + q(x)y = f(x)$$

it is called homogeneous if $f(x) = 0$ and non-homo if $f(x) \neq 0$. Linear means the equation involved the linear combination of $y^{(n)}$.

Theorem 2. ***Principle of superposition for homogeneous equations***
If $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

then so is $y(x) = c_1y_1(x) + c_2y_2(x)$, which is also the ***general solution*** *of the ODE.*

proof needed to be made up which is in the notes

Theorem 3. ***Unique existence*** *Suppose p,q and f are continuous function on the interval I and $x_0 \in I$. Let $y_1, y_0 \in \mathcal{R}$. Then the Second order liner ODE (both homo and non-homo) with initial conditions*

$$y(x_0) = y_0 \;\; \& \;\; y'(x_0) = y_1$$

has a unique solution $y(x)$ on the entire interval I.

What need to be notice is that we need k initial conditions for kth order differential equations.

(a) **The method fo Redution of Order**

If a solution $y_1(x)$ is known for th homo. ODE, then we can find a second solution $y_2(x)$ by proposing

$$y_2(x) = y_1(x) \cdot v(x)$$

It can be shown that $w = v'$ satisfies a first order linear equation which we can solve. This method is general. It can be shown with the coefficient all as function of x. Need to be made up.

(b) **Constant coefficient 2nd linear ODE**

The form of this constant one is simply

$$ay'' + by' + cy = 0$$

where a, b and c are all constant. Here we are motivated by

$$y'' - k^2 y = 0 \quad \longrightarrow \quad y(x) = e^{2x}$$

So we try $y(x) = e^{rx}$. Then the DE becomes

$$(ar^2 + br + c)e^{rx} = 0.$$

which indicates

$$ar^2 + br + c = 0$$

which is called the **characteristic equation.**

Since the char. eq is quadratic so we can use the common method to solve for the roots. It also have three cases for solutions:$b^2 - 4ac >\!<\!= 0$.

For the case $b^2 - 4ac > 0$ it is quite simple. Combining with the theorem above we can find the two solution y_1 and y_2 and consequently the general solution. Finally with the given initial conditions, we find c_1 and c_2.

For the case $b^2 - 4ac = 0$, two roots are the same. Then the general solution become

$$y(x) = c_1 e^{rx} + c_2 \cdot x e^{rx}$$

Then repeat the similar process as above.

For the case $b^2 - 4ac < 0$, we should expect the complex solution which indeed is. No here we need some knowledge of complex number. By the solution of quadratic equations, we have the solution

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

since $b^2 - 4ac < 0$, then we have

$$= \frac{-b \pm \sqrt{4ac - b^2} \cdot i}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i = \lambda \pm i\mu$$

just use the greek letter for a simpler format. Then by the same reason that

$$y_0, y_1 = e^{(\lambda \pm i\mu)x}$$

are solutions for DE, but we prefer real value solutions. So for $y_0 = e^{(\lambda + i\mu)x}$ we have

$$y_a = Re\ y_o = Re\,(e^{\lambda x} \cdot e^{i\mu \cdot x}) = Re[e^{\lambda x}(\cos(\mu\, x) + i \cdot sin(\mu x))] = e^{\lambda x} \cos(\mu x)$$

$$y_b = Im\, y_0 = Im(...) = e^{\lambda x} \sin(\mu x)$$

Also for the case $y_1 = e^{(\lambda - i\mu)x}$. The result is

$$\bar{y}_a = e^{\lambda x} \cos(\mu x) = y_a$$

$$\bar{y}_b = ce^{\lambda x} \sin(\mu x) = -y_b$$

so the general solution become

$$y(x) = c_1 \cdot e^{\lambda x} \cos(\mu x) + c_2 \cdot e^{\lambda x} \sin(\mu x)$$

解释：从向量空间角度理解，$y(x)$ 是 *DE* 的解，即

$$\mathcal{L}y = ay'' + by' + cy = 0$$

即，$\mathcal{L}$ 作为一个 *operator* 使得该 *DE* 等于零。为此，解的实部和虚部必须同时等于零。所以，令 y_α 是 *DE* 的一个解且 *a*,*b* 和 *c* 都是常数，则

$$\mathcal{L}y_\alpha = ay''_\alpha + by'_\alpha + cy_\alpha = 0$$

$$\implies (Re\ \mathcal{L}y_\alpha = 0) \wedge (Im\ \mathcal{L}y_\alpha = 0)$$

$$\implies (\mathcal{L}(Re\, y_\alpha) = 0) \wedge (\mathcal{L}(Im\, y_\alpha) = 0)$$

$$\implies \mathcal{L}(y_1) = 0 \wedge \mathcal{L}(y_2) = 0$$

Complex number relative in this course

(a) **Euler's equation**

The formula for Euler's equation is

$$e^{it} = cos(t) + isin(t)$$

the proof is using Tyler's expansion, omit here make up later. The Euler's identity is where $t = \pi$, then

$$e^{i\pi} = -1$$

another property for complex number used here is

$$e^{a+bi} = e^a \cdot e^{ib} = e^a[cos(b) + i\sin(b)]$$

An application for Euler's equation is to prove the double angle formula.

5. **Mechanical Vibration: Spring-Mass system**

情境介绍：一个质量为 m 的物块被一弹簧链接，固定在左侧墙体上。记起始位置为 0，并以力向右侧抻直一定的距离。则令 $x(t)$ 是一有正负的值，表示相对于起始位置 0 点的距离。对其进行受力分析之后，根据牛顿第二定律可得，

$$(ma =)mx'' = F_{Spring} + F_{Damping} + F_{ExternalForce}$$

here we do not consider external force and ① $F_{Spring} = kx$ ② $F_{Damping} = -cx'$ which is simply the air resistance ③ $F_{ext} = 0$. Then we have

$$F_{Total} = mx'' + cx' + kx$$

本质上是模拟了拉伸后松手时一刻及以后的运动模型。注意 $m, k > 0, c \geq 0$。

(a) **Undamped case: c = 0**

Consider $F = 0$, What does it means for F=0?. The equation becomes

$$mx'' + kx = 0$$

char. eq is $mr^2 + k = 0$, then $r = \pm\omega_0 i$, where $\omega_0 = \sqrt{\frac{k}{m}}$. Then the general solution becomes

$$x(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t)$$

since $e^0 = 1$. x 的表达式可以通过配方变成另一种形式，即

$$Acos\,\omega t + Bsin\omega t = R \cdot cos(\omega t + \delta)$$

where $R = \sqrt{A^2 + B^2}$ and $cos(\delta) = \frac{A}{R}.sin(\delta) = \frac{B}{R}$. Then R is called the amplitude and δ is called the 'phase shift'. 这个被用来计算 amplitude 和 period (Period $= 2\pi/\omega$).

(b) **Damped case: c > 0**

Still here let F_{Total} equals zero. Then the equation becomes

$$mx'' + cx' + kx = 0$$

Our expect (truth): ① 由于能量损失，最后会回到原点。所以 $\lim_{t\to\infty} x(t) = 0$ ②For $0 < c << 1$, small damping, thus slow decaying, solution still oscillates ③ For $c >> 1$, large damping, thus fast decay, solution does not oscillates.

Then if we solve this equation, depending on Δ there are three cases. For $\Delta > 0$ and $\Delta = 0$ it is exactly same as others with the solution

$$y_{\Delta>0} = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}$$

$$y_{\Delta=0} = c_1 \cdot e^{rt} + c_2 \cdot x \cdot e^{rt}$$

and for this two cases the solution does not oscillate (since not trig. terms). For the case $\Delta < 0$, we have

$$y_{\Delta<0} = c_1 \cdot e^{-\frac{c}{2m}} \cos(\omega t) + c_2 \cdot e^{-\frac{c}{2m}} \sin(\omega t)$$

So the cases become: ① $\Delta < 0$ under damping ② $\Delta = 0$ critical damping ③ $\Delta > 0$ overdamping. 注意当 $\Delta = 0$ 即 $c^2 - 4mk = 0$ 时，解出的 $x(t)$ 没有虚部。因此也没有 trig. terms，因此不会 oscillating.

(c) **Non-homogeneous 2nd linear ODE** The form of non-homo 2nd linear ODE is

$$\mathcal{L}y = A(t)y'' + B(t)y' + C(t)y = f(x)$$

Theorem 4.

$$y(t) = y_p(t) + y_c(t)$$

where $y_c = c_1 y_1(t) + c_2 y_2(t)$ is the general solution of $\mathcal{L}y = 0$, which is the solution of its homo. ODE, we call it the complementary homo. solution.

6. **Method of Undetermined Coefficients**

This is the first method to solve y_p for a large class of $\mathcal{L}\,y = f$ where

$$\mathcal{L}y = ay'' + by' + cy = \sum_{n=1}^{N} p_n(t) \cdot e^{\alpha t}(c_1 cos(\beta t) + c_2 sin(\beta t)) = f(t)$$

where $p_n(t)$ is the polynomial w.r.t t. The the solution y_p(the particular solution) should be in the similar form. So we divides the condition if $f(t)$ into different cases to see how to solve for $y_p(t)$ and then combine with its complementary homo. to get the general solution.

上述'in the similar form' 指，比如 $f(t)$ 是 poly. 和 trig. 的乘积，则 y_p 也应该是是 poly. 和 trig. 的乘积；如果 $f(t)$ 是 exp. 和 pol. 的乘积，则 y_p 也应该是 poly. 和 trig. 的乘积。

注意，当有 poly 的时候，假设的 y_p 应当是从其最高次到最低次的 linear combination; 当涉及到 trig. function 时，可能需要是 sin 和 cos 的 linear combination.未完待续

7. **Forced oscillation and resonance**

In this section we consider the 2nd ODE as

$$mx'' + cx' + kx = f(t) = F_0 \cos(\omega t)$$

which we specify the force in periodic form. Similar as before, we discuss in two cases – damped and undamped.

(a) **Undamped case: c = 0**

The ODE becomes

$$m''x + kx = F_0 \cos(\omega t)$$

then we get the $x_c(t)$ is

$$x_c(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency. Then for the particular solution $x_p(t)$ we have

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

then if $\omega \neq \omega_0$, then it is the particular solution with the give I.C solving A and B; if $\omega = \omega_0$, then

$$x_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$$

Combine those two cases together we can see

- when $\omega \neq \omega_0$, amplitude = Bt which is growing in t
- when $\omega = \omega_0$, amplitude = $\frac{F_0}{m|\omega_0^2 - \omega^2|}$, which does not grow in t but get larger and larger as $\omega \to \omega_0$.

This is a phenomenon of resonance（共振）.

(b) **Damped case: c > 0**

The ODE becomes

$$mx'' + cx' + kx = F_0 cos(\omega t)$$

Then we figure out the case for $x_c(t)$ is one of the following:

- when $\Delta > 0$
$$x_c(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
- when $\Delta = 0$
$$x_c(t) = Ae^{-\frac{c}{2m}t} + Bte^{-\frac{c}{2m}t}$$
- when $\Delta < 0$
$$x_c(t) = ae^{-\frac{c}{2m}t}\cos(\mu t) + be^{-\frac{c}{2m}t}\sin(\mu t)$$
where $\mu = \frac{1}{2m}\sqrt{4mk - c^2}$.

Then the form of a particular solution is

$$x_p(t) = Acos(\omega t) + Bsin(\omega t)$$

NO OVERLAP WITH $x_c(t)$ AT ALL, hence valid. Then combine those two cases together we have

$$x_g(t) = \underbrace{x_c(t)}_{Exp.decay} + \underbrace{x_p(t)}_{Peri.persistent}$$

so as $t \to \infty$ x_c is negligible as the transient part and $x_p(t)$ still exists as the periodic part. 注意, the long time behaviour = steady periodic part 是由于 the given periodic forcing, 与 initial condition 无关。Notice some times we may use matrix to solve the parameter A and B.

8. **Laplace transform**

Definition 3. For a given function $f(t)$ defined for $t > 0$, its Laplace transform is another function $\mathcal{L}\{(s)$ defined by

$$\mathcal{L}f(x) = \int_0^\infty f(t)e^{-st}\, ds$$

where s is a real parameter in the improrper integral.

Recall the definition for a convergence in improper integral, which is the limit for

$$\int_a^\infty g(t)\,dt = \lim_{A\to\infty}\int_a^A g(t)\,dt$$

exists for all A, otherwise it diverges.

Remark 1. If $|g(t)| \leq h(t)$ and $\int_0^\infty h(t)\,dt$ converges, then $\int_0^\infty g(t)dt$ converges.

Notice, for the Laplace transform, the larger the S, the smaller the integrand, the more likely to converge. The domain of $\mathcal{L}f(x)$ is the **set of s that makes the integral converges**. It is usually an open interval (a, ∞) for some a.

(a) **Properties of Laplace transform**

i. It is a linear map (operator) which satisfies

$$\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}f + c_2\mathcal{L}g$$

ii. Not multiplicative, which is

$$\mathcal{L}\,f \cdot \mathcal{L}\,g \;\neq\; \mathcal{L}\{fg\}$$

iii. Uniqueness question

(b) **Inverse Laplace transform**

Simply defined as the inverse of Laplace transform. If $\mathcal{L}\{f(t)\} = F(s)$, then we define

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

(c) **First shifting property**

Definition 4. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{-at}\cdot f(t)\} = F(s+a)$$

Proof.

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty f(t)e^{-at}\cdot e^{-st}\,dt = \int_0^\infty f(t)e^{-t(s+a)}dt = F(s+a)$$

□

Also, the inverse also satisfy s.t $\mathcal{L}^{-1}\{F(s+a)\} = e^{-at}f(t)$.

(d) **Laplace transform of derivatives and ODEs**

Lemma 1.

$$\mathcal{L}\{f'\} = s\cdot\mathcal{L}\{f\} - f(0)$$

and for f'', consequently we have

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

This lemma can be used to solve the ODE.

Lemma 2. *(The second shifting law)*
Let $a \geq 0$. Then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as} \cdot \mathcal{L}\{f(t)\}$$

The proof is simply using integral by substitution.

(e) **Heaviside Function 单位阶跃函数**

Definition 5. The Heaviside function is defined as

$$H(x) = \begin{cases} 0 \ , \ x < 0 \\ 1 \ , \ \ x > 0 \end{cases}$$

The middle point at $x = 0$ is not important. 单位跃阶函数用来计算有断点的 step function 的拉普拉斯变换 (piecewise continuous function).

Example: Find the L.T of the $u(t-a)$ and $f(t) = 1$ if $x \in (a, b)$ and 0 otherwise.

Answer:

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\,dt = \int_a^\infty e^{-st}\,dt = -\frac{e^{-st}}{s}\Bigg|_a^\infty = \frac{e^{-sa}}{s}$$

Then for $f(t)$ we can rewrite the function into $f(t) = u(t-a) - u(t-b)$ then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-a)\} - \mathcal{L}\{u(t-b)\} = \frac{e^{-sa} - e^{-sb}}{s}$$

Example:

Answer:

9. **Convolution 卷积**

Definition 6. The convolution of the function f and g is defined as

$$f * g = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau$$

which is equivalent to

$$f * g = \int_{-\infty}^{\infty} g(t-\tau)f(\tau)\,d\tau$$

which is commutativity which can be proved by change of variable. In 215 we assume the function f and g supports only on $[0, \infty)$, so the integral above supports only on $[0, t]$ which is

$$f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

The convolution has following properties: ① $f * g = g * f$ ② $(cf) * g = c(f * g) = f * cg$ ③ $(f * g) * h = f * (g * h)$.

Theorem 5.

$$\mathcal{L}\{f * g\} = F(s) \cdot G(s)$$

The proof simply involves double integral and using the change of variable.

10. **Dirac delta function and Impulse response**

The somewhat formal definition is

$$\delta(t) = \lim_{\epsilon \to 0} d_\epsilon(t) \;\equiv\; \delta(t) = \begin{cases} \infty \;, \; t = 0 \\ 0 \;, \;\; t \neq 0 \end{cases}$$

and it should satisfy

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

此处应该指出的是，如果给定区间内包含 0，则积分结果等于 1；如果区间不包含零则积分结果为 0。另外，对于任意的连续函数 $f(t)$,delta 函数满足

$$\int_a^b f(t)\delta(t)\, dt = f(0)$$

we can define $\delta(t)$ rigorously as the linear map: $f(t) \mapsto f(0)$ question. Translate the rectangle to $d_\epsilon(t-a) \to \delta(t-a)$. 未完，先记住结论

$$\delta(t-a) = \frac{d}{dt} u(t-a)$$

then the laplace transform is

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

11. **First order systems of DE**

The general form of a fist order DE system is

$$\frac{d}{dt}\overrightarrow{x} = P(t)\overrightarrow{x} + \overrightarrow{g}(t)$$

where P(t) is a matrix. The system is said to be linear in x if

$$F_j(t, x_1, ..., x_n) = g_j(t) + p_{j1}(t)x_1 + p_{j2}(t)x_2 + ... + p_{jn}(t)x_n, \;\; j \in [1, n]$$

where j is the index of the jth equation and n is the nth variable x.

(a) **Solution Space**

Let V be the set of all solution of a homogeneous system $\overrightarrow{x}' = P(t)\overrightarrow{x}$. Then the solution space is

$$V = \{\overrightarrow{x}(t) : \overrightarrow{x}' = P(t)\overrightarrow{x}, t \in (a, b)\}$$

As a vector space which consisting of all $x(t)$s' satisfying the equation, any linear combination of elements in it is also a solution.

Now consider the non-homo system.

Theorem 6. *If $\overrightarrow{x}_p(t)$ is a particular solution of*

$$\frac{d}{dt}\overrightarrow{x}(t) = P(t)\overrightarrow{x}(t) + \overrightarrow{g}(t)$$

then every solution can be written as

$$\overrightarrow{x}(t) = \overrightarrow{x}_c(t) + \overrightarrow{x}_p(t)$$

(b) **Fundamental matrix**

For homogeneous system, let $\overrightarrow{x}_1(t)$, $\overrightarrow{x}_2(t)$ be two linearly independent solution to the system. Then we define the matrix

$$\underline{X} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

as fundamental matrix consisting of the column vector as $\overrightarrow{x}_1$ and $\overrightarrow{x}_2$.

12. **Eigenvalue Method for Homo. Constant coefficient system**

Still consider the sysytem $\frac{d}{dt}\overrightarrow{x} = Ax$ where A is constant real n by n matrix. The solution space is a n-dimensional vector space. We want to find a simple basis of V. We try $\overrightarrow{x} = e^{\lambda t}\overrightarrow{v}$, where $\overrightarrow{v}$ is a constant vector. We find

$$\lambda e^{\lambda t}\overrightarrow{v} = A\, e^{\lambda t}\overrightarrow{v} \implies \lambda\overrightarrow{v} = A\overrightarrow{v}$$

hence $\overrightarrow{v}$ is a eigenvector of A with eigenvalue λ. So our strategy is to find all possible eigenvalues: real and distinct, repeated, complex. 仍然通过之前的方法找到 ODE 的解，然后 transfer 到 matrix 中。

(a) **Complex eignvalue**

Lemma 3. *If a real matrix A has an eigenvalue λ with eigenvector v, then it also have a eigenvalue $\bar{\lambda}$ with corresponding eigenvector $\bar{v}$ which is simply the conjugate of v.*

Lemma 4. *If $\overrightarrow{x} = y(t) + iz(t)$ is a complex valued solution to $\frac{d}{dt}\overrightarrow{x}$ where A is real. Then $y(t)$ and $z(t)$ are also real valued solutions.*

(b) **Repeated Eigenvalues**

Algebraic multiplicity of an eigenvalue is power m of λ in the char. equation $(\lambda - \lambda_n)^m$; and the geometric multiplicity is the maximum number of linearly independent eigenvector of λ_n. Notice, the *geo.multi.* is always less than or equal to the *algb.multi.*.

Remark 2. In general, if λ_1 is a repeated eigenvalue(alg. multiplicity greater than 1) of matrix A with only one eigenvector v_1, in addition to $\overrightarrow{x}(t) = e^{\lambda_1 t}\overrightarrow{v}_1$, the second solution is in form of

$$\overrightarrow{x}_2(t) = e^{\lambda_1 t}(t\overrightarrow{v}_1 + \overrightarrow{v}_2)$$

since we need

$$\overrightarrow{x}'_2 = A\overrightarrow{x}_2$$

we can get

$$(A - \lambda_1 I)\overrightarrow{v}_2 = \overrightarrow{v}_1$$

use this to find $\overrightarrow{v}_2$. For example, we have already known A has two egi. value $\lambda_1, lambda_2$ and λ_2 is repeated. Then

$$\overrightarrow{x}_c(t) = c_1 e^{\lambda_1 t}\overrightarrow{v}_1 + c_2 e^{\lambda_2 t}\overrightarrow{v}_2 + c_3 e^{\lambda_2 t}(t\overrightarrow{v_1} + \overrightarrow{v}_2)$$

13. **Phase portrait for 2D linear system**

Consider $\overrightarrow{x}(t) \in \mathcal{R}^2$ and $A \in M(2\times 2, \mathcal{R})$. Each solution to $\overrightarrow{x}' = A\overrightarrow{x}$ form a trajectory in $\mathcal{R}^2$. Notice the trajectory of $y(t) = \overrightarrow{x}(t_C)$ is the same as $\overrightarrow{x}(t)$. In phase portrait we consider all trajectory in $\mathcal{R}^2$. We have 3 cases: $\lambda_1 < \lambda_2$ real unique, $\lambda_1 = \bar{\lambda_2}$ complex conjugate and $\lambda_1 = \lambda_2$ repeated.

Case1(a): $\lambda_1 < \lambda_2 < 0$ (sink)

Case1(b): $0 < \lambda_1 < \lambda_2$ (source)

Case1(a): $\lambda_1 < 0 < \lambda_2$ (saddle)

Case2(b): $\lambda_1, \lambda_2 = a + bi, 0 < a < b$ (spiral source)

Case2(c): $\lambda_1, \lambda_2 = a + bi, a < 0 < b$ (spiral sink)

Case3: $\lambda_1 = \lambda_2 \neq 0$ (hard to draw)

14. **Nonhomo. System**

The form is simply

$$\frac{d}{dt}\overrightarrow{x} = P(t)\overrightarrow{x} + \overrightarrow{f}(t)$$

recall fundamental matrix, it satisfies

$$\underline{\overline{X}}' = P\underline{\overline{X}}$$

(a) **Variation of Parameter method**

$$\overrightarrow{x}_t = \int^t \underline{\overline{X}}^{-1}(s) f(s)\, ds$$

this is the most general method to solve the solution, even though the matrix is time dependent.

(b) **Undetermined coefficient**

Exactly the same as the previous case. We first solve the homo case and guess the particular solution corresponding to the term $\overrightarrow{f}$ and also check whether it is overlapping with the homo solution.

15. **Non linear system**

Here we just talk about the autonomous non linear system, which is

$$\frac{d}{dt}\overrightarrow{x} = \overrightarrow{F}(x, y)$$

where F does not depends on t containing just x and y. Notice $\overrightarrow{x} = (x, y)^T$. Critical point = equilibrium = fixed point and it means, let $\overrightarrow{x}_0$ is a fixed point then

$$\overrightarrow{F}(\overrightarrow{x}_0) = 0$$

(a) **Linearization in two dimensional case**

We focus one the behavior around the fixed point, so we find the linearization by Taylor expansion (two dimension). Let $p_0 = (x_0, y_0)$ be the fixed point then

$$f(x, y) = f(x_0, y_0) + f_x(p)(x - x_0) + f_y(p)(y - y_0) + h.o.t$$

$$g(x, y) = f(x_0, y_0) + g_x(p)(x - x_0) + g_y(p)(y - y_0) + h.o.t$$

hot means higher order term. Then we use the jacobin matrix

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + h.o.t$$

in short

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = J(p)\begin{bmatrix} u \\ v \end{bmatrix}$$

where $u = x - x_0$ and $v = y - y_0$.